OPERATIONS IN THE ALGEBRAIC K-THEORY OF SPACES

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The purpose of this note is to show that the analogue of the Kahn-Priddy theorem is valid for the algebraic K-theory of spaces.

To make this more precise we first recall the Kahn-Priddy theorem in a convenient form, and introduce some notation. Let $Q(X)$ denote the unreduced stable homotopy of X, $Q(X) = \Omega^\infty S^\infty(X_+)$. Let $\tilde{Q}(X)$ be the reduced part, we think of it as $\mathrm{fibre}(Q(X) \to Q(*))$, the homotopy theoretic fibre; here $*$ denotes a one-point space. Let Σ_n denote the symmetric group on n letters, and $B\Sigma_n$ its classifying space. Associated to the universal covering map

$$* \simeq E\Sigma_n \longrightarrow B\Sigma_n$$

there is a *transfer map*

$$Q(B\Sigma_n) \longrightarrow Q(*) .$$

By composition with $\tilde{Q}(B\Sigma_n) \to Q(B\Sigma_n)$ one obtains from it a map $\tilde{Q}(B\Sigma_n) \to Q(*)$. Let p be a prime and let the subscript (p) denote localization at p. The Kahn-Priddy theorem may then be formulated to say that the map of localized homotopy groups

$$\pi_j\tilde{Q}(B\Sigma_p)_{(p)} \longrightarrow \pi_j Q(*)_{(p)}$$

is surjective for $j > 0$.

Let $A(X)$ denote the algebraic K-theory of X (cf. [9] or [5]; it will be reviewed below), and let $\tilde{A}(X) = \mathrm{fibre}(A(X) \to A(*))$ be the reduced part. The analogue of the Kahn-Priddy theorem to be proved here, says that for any prime p the transfer map

$$\pi_j\tilde{A}(B\Sigma_p)_{(p)} \longrightarrow \pi_j A(*)_{(p)}$$

is surjective for $j > 0$.

As with Segal's proof of the Kahn-Priddy theorem [6] this result will be deduced from the existence of certain operations. These operations may be regarded as extensions of the power operations θ^n which Segal constructed in stable homotopy theory. At any rate, the relation is so close that it seems appropriate to use the same name.

Theorem. There are maps $\theta^n: A(*) \to A(B\Sigma_n)$ which satisfy

(1) θ^1 = identity map

(2) The combined map

$$\theta = \Pi_{n\geq 1} \theta^n : A(*) \longrightarrow \Pi_{n\geq 1} A(B\Sigma_n)$$

is a map of H-spaces if the right hand side is equipped with the H-space structure arising from the juxtaposition pairings $A(B\Sigma_m) \times A(B\Sigma_n) \longrightarrow A(B\Sigma_{m+n})$.

(3) The composite of $\theta^n: A(*) \to A(B\Sigma_n)$ with the transfer map $\phi_n: A(B\Sigma_n) \to A(*)$ is the same (up to weak homotopy) as the polynomial map from $A(*)$ to itself given by the integral polynomial

$$p_n(x) = x(x-1)\dots(x-n+1) .$$

Property (3) refers to the fact that $A(*)$ is a 'ring' - it will certainly suffice here to know that the homotopy functor represented by $A(*)$ has a ring structure. Thus given a homotopy class of maps $f \in [Y,A(*)]$, and an integral polynomial $p(x)$, one can evaluate $p(x)$ on f . The map in property (3) is obtained in this way by evaluating the polynomial $p_n(x)$ on the identity map of $A(*)$.

To apply the theorem, we note if Y is a suspension, and $f \in [Y,A(*)]$ is the homotopy class of a map having its image in the connected component of zero, then $f^2 = 0$. (For f^2 may be represented by the product of a pair of maps which take, respectively, the upper and lower hemisphere into zero). In particular, this remark applies to $f \in \pi_j A(*)$ if $j > 0$ (we may dispose of basepoints in view of the (additive) H-space structure on $A(*)$). It follows that, for $j>0$, the endomorphism of $\pi_j A(*)$ induced by the polynomial $p_n(x)$ is the same as that induced by its linear term $(-1)^{n-1}(n-1)!x$. Applying the theorem, we obtain that the map

$$(\phi_n\theta^n)_* : \quad \pi_j A(*) \longrightarrow \pi_j A(*) , \quad j > 0 ,$$

is given by multiplication with $(-1)^{n-1}(n-1)!$.

We specialize to the case where $n = p$ is a prime. Then $(-1)^{p-1}(p-1)!$ is a unit modulo p , so it follows that the transfer map

$$(\phi_p)_* : \quad \pi_j A(B\Sigma_p) \longrightarrow \pi_j A(*) , \quad j > 0 ,$$

is surjective modulo p . Now

$$\pi_j A(B\Sigma_p) \approx \pi_j \tilde{A}(B\Sigma_p) \oplus \pi_j A(*)$$

and (cf. lemma 1.3 below) the composite map

$$\pi_j A(*) \longrightarrow \pi_j A(B\Sigma_p) \xrightarrow{\text{transfer}} \pi_j A(*)$$

is given by multiplication with the order of Σ_p , which is 0 modulo p . It follows that the composite map

$$\pi_j\widetilde{A}(B\Sigma_p) \longrightarrow \pi_j A(B\Sigma_p) \xrightarrow{\text{transfer}} \pi_j A(*) \ , \quad j > 0 \ ,$$

is still surjective modulo p .

We can conclude with an application of Nakayama's lemma. In fact, Dwyer has shown [1] that $\pi_j A(*)$ is finitely generated. So Nakayama's lemma applies, showing that the map of localizations

$$\pi_j\widetilde{A}(B\Sigma_p)_{(p)} \longrightarrow \pi_j A(*)_{(p)} \ , \quad j > 0 \ ,$$

will be surjective as soon as its reduction modulo p is. This we have just seen.

It remains to prove the theorem. The construction of the operations θ^n along with the verification of their properties will be given in section 2. The method is that of Segal, essentially. Briefly, Segal's construction is concerned with sets and their isomorphisms (the formulation in [6] is on the represented functor level, in terms of covering spaces) whereas we have to work here with the larger category of simplicial sets and their weak homotopy equivalences. The characteristic feature of the method is that the construction is done first on an elementary level (by explicit manipulation of sets, resp. simplicial sets) and is then extended quite indirectly by appeal to a certain universal property. In the present paper, the discussion of that universal property, together with a certain amount of background material, makes up the preliminary section 1. At the end of the paper there are some remarks on generalizations and variants of the construction.

1. Review of $A(X)$. Let $R(*)$ denote the category of pointed simplicial sets, and $R_f(*)$ the subcategory of those Y which are *finite* (that is, generated by finitely many simplices; equivalently, the geometric realization $|Y|$ is compact). The category $R_f(*)$ comes equipped with two distinguished subcategories, the category of *cofibrations* (injective maps) on the one hand, and the category of *weak homotopy equivalences* on the other; the latter category will be denoted $hR_f(*)$.

Taking this situation as a model one defines the notion of a *category with cofibrations and weak equivalences*. This is a category C pointed by a zero object and equipped with subcategories $co(C)$ and $w(C)$ whose morphisms are called *cofibrations*, resp. *weak equivalences*, and where certain simple properties of a formal nature are required to hold, essentially the possibility of 'gluing' (cobase change by cofibration) and the validity of the 'gluing lemma' for the weak equivalences.

It is possible in this situation to write down a certain simplicial category $wS.C$. The category wS_nC (i.e. the category in degree n of this simplicial category) has as its objects the filtered objects (sequences of cofibrations) of length n ,

$$Y_{o1} \rightarrowtail Y_{o2} \rightarrowtail \ \cdots \ \rightarrowtail Y_{on} \ ,$$

and the morphisms are the weak equivalences of filtered objects, that is, the natural transformations

$$\begin{tikzcd} Y_{o1} \arrow[r, tail] \arrow[d, "\wr"] & Y_{o2} \arrow[r, tail] \arrow[d, "\wr"] & \cdots \arrow[r, tail] & Y_{on} \arrow[d, "\wr"] \\ Y'_{o1} \arrow[r, tail] & Y'_{o2} \arrow[r, tail] & \cdots \arrow[r, tail] & Y'_{on} \end{tikzcd} .$$

(There is a little technical point. The simplicial structure involves quotient objects

$$Y_{ij} \approx Y_{oj}/Y_{oi} \quad (= Y_{oj} \cup_{Y_{oi}} 0)$$

which therefore better be well-defined, not just well-defined up to isomorphism (they exist by assumption). For this reason one blows up the category to a larger but equivalent category by including such choices in the data. Cf. [9] or [5] for details.)

In the basic case of the category $R_f(*)$ one defines $A(*)$ as the loop space of the geometric realization of this simplicial category,

$$A(*) = \Omega|hS.R_f(*)| \ .$$

As a general remark let us note that the equivalence of categories

$$wS_1C \sim w(C)$$

(together with the fact that wS_oC is the trivial category containing only 0 , the basepoint) gives rise to an inclusion of the suspension,

$$\Sigma|w(C)| \longrightarrow |wS.C| \ .$$

Passing to the adjoint we obtain a map

$$|w(C)| \longrightarrow \Omega|wS.C| \ .$$

In particular we obtain in this way a map

$$|hR_f(*)| \longrightarrow A(*) \ .$$

On the level of connected components this map is essentially the Euler characteristic, cf. [9]. In the following two lemmas we show how the map can be used to characterize $A(*)$ in terms of a universal property.

We shall denote by

$$s,\ t,\ q \ : \ S_2R_f(*) \longrightarrow R_f(*)$$

the maps which to a cofibration sequence $Y_{o1} \rightarrowtail Y_{o2} \twoheadrightarrow Y_{12}$ associate its *subobject*, *total object*, and *quotient object*, respectively. Let svq denote the map given by the sum of s and q .

Lemma 1.1. The two composite maps

$$|hS_2R_f(*)| \overset{t}{\underset{s\vee q}{\rightrightarrows}} |hR_f(*)| \longrightarrow A(*)$$

are homotopic.

Proof. The *additivity theorem* (cf. [9], [5], and [11]) says that if C is a category with cofibrations and weak equivalences then so is S_2C and the map

$$wS.(S_2C) \xrightarrow{wS.(s) \times wS.(q)} wS.C \times wS.C$$

is a homotopy equivalence. An immediate consequence is that the section of this map is also a homotopy equivalence, the section is the map which takes Y_{01} and Y_{12} to the split cofibration sequence $Y_{01} \rightarrowtail Y_{01} \vee Y_{12} \twoheadrightarrow Y_{12}$. This in turn implies that the two maps

$$wS.(t) \ , \ wS.(s\vee q) : \quad wS.(S_2C) \longrightarrow wS.C$$

are homotopic, for they agree on split cofibration sequences. We have thus established that in the diagram

$$\begin{tikzcd} |wS_2C| \arrow[r, shift left, "t"] \arrow[r, shift right, "s\vee q"'] \arrow[d] & |wC| \arrow[d] \\ \Omega|wS.(S_2C)| \arrow[r, shift left] \arrow[r, shift right] & \Omega|wS.C| \end{tikzcd}$$

the two composite maps through the lower left are homotopic. The diagram becomes commutative if we discard the upper (resp. lower) arrow from both the upper and lower row. It results that the two composite maps through the upper right are homotopic. In the case where C is $R_f(*)$ this is the assertion of the lemma. □

Lemma 1.2. Let F be a representable abelian-group-valued homotopy functor on the category of finite CW complexes. Let

$$\Phi : \quad [\ \ , |hR_f(*)|\ ] \longrightarrow F$$

be a map of semigroup-valued functors, and suppose that Φ equalizes the two maps

$$[\ \ , |hS_2R_f(*)|\ ] \overset{t_*}{\underset{(s\vee q)_*}{\rightrightarrows}} [\ \ , |hR_f(*)|\ ] \ .$$

Then there exists a unique map of abelian-group-valued functors

$$\Phi' : \quad [\ \ , A(*)\ ] \longrightarrow F$$

having the property that for every n the diagram

$$\begin{tikzcd}
{[\ ,|h\mathcal{R}_f(*)|]} \arrow[d] & {[\ ,|h\mathcal{R}^n(*)|]} \arrow[l, "j_*"'] \arrow[r, "j_*"] & {[\ ,|h\mathcal{R}_f(*)|]} \arrow[d, "\Phi"] \\
{[\ ,A(*)\ ]} \arrow[rr, "\Phi'"] & & F
\end{tikzcd}$$

commutes, where $h\mathcal{R}^n(*)$ denotes the union of connected components of $h\mathcal{R}_f(*)$ given by wedges of n-dimensional spheres, and the map j_* is induced by the inclusion map $h\mathcal{R}^n(*) \to h\mathcal{R}_f(*)$.

Remark. The commutativity of those diagrams serves to force the uniqueness of Φ' . A neater statement would be to simply say that Φ' extends Φ . However it is not clear if this is true.

Proof. We indicate how $A(*)$ may be re-expressed in terms of the categories $h\mathcal{R}^n(*)$ by the *group completion* construction. It will then be possible to simply quote a result from that context. We use Segal's version of group completion [7].

The category $h\mathcal{R}_f(*)$ has a composition law induced from the coproduct on the ambient category $\mathcal{R}_f(*)$. It can therefore be considered as the *underlying category of a* (special) *Γ-category*, and one can form the *nerve* of that Γ-category, a certain simplicial category $N_\Gamma(h\mathcal{R}_f(*))$, cf. [7] for the construction, and e.g. [8] for a detailed discussion of it. Briefly, the category in degree n of $N_\Gamma(h\mathcal{R}_f(*))$ is equivalent to the product category $(h\mathcal{R}_f(*))^n$; an object consists of a tuple Y_{01}, $Y_{12},\dots,Y_{n-1,n}$ plus all kind of choices related to the sum in $\mathcal{R}_f(*)$; for example the data include the choice of an object $Y_{01}\vee Y_{12}$ together with maps $Y_{01} \to Y_{01}\vee Y_{12} \leftarrow Y_{12}$ expressing the object as a sum, other data are implied by these, for example a projection $Y_{01}\vee Y_{12} \to (Y_{01}\vee Y_{12})/Y_{01} \approx Y_{12}$.

There is a map of simplicial categories

$$N_\Gamma(h\mathcal{R}_f(*)) \longrightarrow hS.\mathcal{R}_f(*) ,$$

it is the forgetful map which in degree n takes

$$(\ Y_{01},\ Y_{12},\ \dots\ ,\ Y_{n-1,n}\ ,\quad \text{choices}\ )$$

to

$$(\ Y_{01} \rightarrowtail Y_{01}\vee Y_{12} \rightarrowtail \dots \rightarrowtail Y_{01}\vee Y_{12}\vee\dots\vee Y_{n-1,n}\ ,\quad \text{(fewer) choices}\ )\ .$$

One can similarly form a simplicial category $N_\Gamma(h\mathcal{R}^n(*))$, and the inclusion $\mathcal{R}^n(*) \to \mathcal{R}_f(*)$ induces one

$$N_\Gamma(h\mathcal{R}^n(*)) \longrightarrow N_\Gamma(h\mathcal{R}_f(*))\ .$$

Composing with the map above we obtain a map

$$N_\Gamma(hR^n(*)) \longrightarrow hS.R_f(*) \ .$$

For varying n these maps are compatible by means of suspension, the diagrams

$$\begin{tikzcd} N_\Gamma(hR^n(*)) \arrow[r] \arrow[d, "\Sigma"] & hS.R_f(*) \arrow[d, "\Sigma"] \\ N_\Gamma(hR^{n+1}(*)) \arrow[r] & hS.R_f(*) \end{tikzcd}$$

commute. Thus there results a map in the limit,

$$\varinjlim_n N_\Gamma(hR^n(*)) \longrightarrow \varinjlim_{(\Sigma)} hS.R_f(*) \ .$$

A basic result now asserts that this map is a homotopy equivalence [11].

Up to homotopy, the term on the right is $hS.R_f(*)$ again. To see this, it suffices to know that the self map Σ of $hS.R_f(*)$ given by the suspension, is a homotopy equivalence. There is a cofibration sequence of functors on $R_f(*)$,

$$\text{identity} \rightarrowtail C \twoheadrightarrow \Sigma$$

where C denotes the cone functor. By the additivity theorem (cf. the proof of lemma 1.1) this implies a homotopy of the induced maps on $hS.R_f(*)$,

$$\text{id} \vee \Sigma \ \simeq \ C \ .$$

As the cone map is nullhomotopic it follows that the suspension map is a homotopy-inverse with respect to the additive H-space structure, in particular therefore it is a homotopy equivalence. $A(*)$ has thus been re-expressed by 'group completion' as

$$A(*) \ \simeq \ \Omega|\varinjlim_n N_\Gamma(hR^n(*))| \ .$$

Suppose now that F is a representable abelian-group-valued homotopy functor, and

$$\Phi : [\ \ , |hR_f(*)|] \longrightarrow F$$

a map of semigroup-valued functors, as in the lemma. By hypothesis Φ converts cofibration sequences into sums. Applying this to the cofibration sequence $\text{id} \rightarrowtail C \twoheadrightarrow \Sigma$, we obtain that $\Phi + \Phi\Sigma_*$ is the zero map, in other words $\Phi\Sigma_* = (-1)\Phi$. Let us define a map

$$\Psi_n : [\ \ , |hR^n(*)|] \longrightarrow F$$

as the composite

$$[\ \ , |hR^n(*)|] \longrightarrow [\ \ , |hR_f(*)|] \xrightarrow{\Phi} F \ ,$$

multiplied by $(-1)^n$. Then $\Psi_n = \Psi_{n+1}\Sigma_*$, so we can obtain a map in the limit

$$\Psi : [\ \ , |\varinjlim_n hR^n(*)|] \longrightarrow F \ .$$

This is a map of semigroup-valued functors, hence, thanks to Segal [7], it factors through a unique map of abelian-group-valued functors

$$[\ \ ,\Omega|N_\Gamma(\varinjlim_n hR^n(*))|] \longrightarrow F$$

that is, through a map

$$\Phi' : [\ \ ,A(*)] \longrightarrow F$$

since $N_\Gamma(\varinjlim hR^n(*)) \approx \varinjlim N_\Gamma(hR^n(*))$.

In view of its construction, the map Φ' extends the map

$$[\ \ ,|hR^n(*)|] \longrightarrow [\ \ ,|hR_f(*)|] \longrightarrow F ,$$

at least for even n . For odd n it must be checked if there is a sign left. There is not. For the diagram

$$\begin{tikzcd} hR^n(*) \arrow[r] \arrow[d, "\Sigma"] & hR_f(*) \arrow[dr] & \\ & & A(*) \\ hR^{n+1}(*) \arrow[r] & hR_f(*) \arrow[ur] & \end{tikzcd}$$

commutes only up to application of Σ to $A(*)$, in other words, the diagram commutes up to homotopy and composition with the homotopy-inverse on $A(*)$. This gives another sign which cancels the former. □

In order to define $A(X)$ for simplicial sets X in general, one uses the category $R(X)$ of the *retractive spaces over* X ; the objects are the triples (Y,r,s) where $r:Y \to X$ is a retraction and s a section of r . The appropriate *finiteness* condition to use is that Y be generated by X together with finitely many additional simplices. The category $R_f(X)$ of the finite objects in $R(X)$ is a category with cofibrations and weak homotopy equivalences, and one defines

$$A(X) = \Omega|hS.R_f(X)| .$$

For some purposes it is useful to know that the finiteness condition may be relaxed to a condition of finiteness up to homotopy, replacing $R_f(X)$ by a somewhat larger category $R_{hf}(X)$. This leads to the same $A(X)$, up to homotopy [11]. (In fact, one could even use spaces dominated by finite ones, in a suitable sense. This would replace the group of connected components (the integers) by a larger group (a suitable projective class group), but it would not alter the higher homotopy groups).

The category $R(X)$ is a framework for studying what may be called the *equivariant homotopy theory parametrized by* X . Another framework for studying that same theory is the category of simplicial sets with G-action where G is a *loop group* for X , that is, a simplicial group such that there exists a universal G-bundle over X, a principal G-bundle $E(G,X)$ with (weakly) contractible total space. Such a G always exists for connected X; for example, Kan [4] has given a construction which is functorial for connected pointed X.

Let $\mathcal{U}(G)$ denote the category of pointed simplicial sets with G-action. The finiteness condition is somewhat delicate, it involves a freeness condition. By definition, $\mathcal{U}_f(G)$ is the subcategory of those G-simplicial sets which are free (in the pointed sense) and finitely generated over G. It is a category with cofibrations and weak homotopy equivalences, so $\Omega|hS.\mathcal{U}_f(G)|$ is defined. One shows this recovers $A(X)$, up to homotopy, if G is a loop group of X. In fact [11], if E is a universal G-bundle over X then an explicit homotopy equivalence is given by the map

$$\begin{aligned} hS.\mathcal{R}_f(X) &\longrightarrow hS.\mathcal{U}_f(G) \\ (Y,r,s) &\longmapsto (Y \times_X E) \cup_X * \,. \end{aligned}$$

A homotopy equivalence in the other direction can also be explicitly described [11], it is the map $hS.\mathcal{U}_f(G) \to hS.\mathcal{R}_{hf}(X)$ which takes a G-simplicial set Z to the associated bundle $* \times^G (Z \times E)$ (=space of orbits for the diagonal action).

Let us discuss maps now. If $(\mathcal{C},\ldots)$ and $(\mathcal{C}',\ldots)$ are categories with cofibrations and weak equivalences, a functor $\mathcal{C} \to \mathcal{C}'$ will be called *exact* if it preserves all the relevant structure. Such a functor induces a map of the associated simplicial categories. For example, the above homotopy equivalences were of this type.

Two kinds of maps on $A(X)$ arise in this way. A map $X \to X'$ induces an exact functor $\mathcal{R}(X) \to \mathcal{R}(X')$ by taking (Y,r,s) to $(X' \cup_X Y,\ldots)$, this restricts to an exact functor $\mathcal{R}_f(X) \to \mathcal{R}_f(X')$, hence induces $A(X) \to A(X')$. On the other hand, if $\tilde{X} \to X$ is the projection of a fibre bundle, it induces an exact functor $\mathcal{R}(X) \to \mathcal{R}(\tilde{X})$ by pullback. If the fibre is finite (resp. finite up to homotopy) the functor restricts to an exact functor from $\mathcal{R}_f(X)$ to $\mathcal{R}_f(\tilde{X})$ (resp. $\mathcal{R}_{hf}(\tilde{X})$) and hence it induces a map $A(X) \to A(\tilde{X})$ called the *transfer*.

Let us note as an aside that the transfer on $A(X)$ provides yet another way for constructing a transfer in stable homotopy. For the stable homotopy of X is a retract of $A(X)$, cf. [10], so a transfer may be defined as the composite map

$$Q(X) \longrightarrow A(X) \longrightarrow A(\tilde{X}) \longrightarrow Q(\tilde{X}) \,.$$

Returning to $A(X)$, we want to know that upon translation into the context of spaces with group action, the two maps described correspond to the usual 'induction' and 'restriction' maps, respectively. Concerning induction this is checked in [11]. Let us check here that restriction corresponds to the transfer. Suppose then that G is a simplicial group and H a simplicial subgroup such that the simplicial set of orbits $* \times^H G$ is finite, up to homotopy. Let EG be any contractible simplicial set on which G acts freely, for example the diagonal simplicial set of $[n] \mapsto G^{n+1}$.

Then EG is a universal G-bundle over the simplicial sets of orbits $* \times^G EG$, so, as mentioned above, the associated-bundle construction gives an exact functor

$$\begin{array}{rcl} \mathcal{U}(G) & \longrightarrow & \mathcal{R}(* \times^G EG) \\ M & \longmapsto & * \times^G (M \times EG) \end{array}$$

inducing a homotopy equivalence $hS.\mathcal{U}_f(G) \to hS.\mathcal{R}_{hf}(* \times^G EG)$. But EG may also be considered as a universal H-bundle over $* \times^H EG$ or what is the same thing, $(* \times^H G) \times^G EG$, and there is a commutative diagram

$$\begin{tikzcd} \mathcal{U}(H) \arrow[r] & \mathcal{R}((* \times^H G) \times^G EG) \\ \mathcal{U}(G) \arrow[r] \arrow[u] & \mathcal{R}(* \times^G EG) \arrow[u] \end{tikzcd}$$

where the arrow on the left is the forgetful map given by the restriction of the action from G to H, and the arrow on the right is the pullback. Thus restriction corresponds to the transfer. (Note we are admitting here [11] that the category of finite objects $\mathcal{U}_f(H)$ may be enlarged to a category $\mathcal{U}_{hf}(H)$ of objects which are finite up to homotopy).

<u>Lemma</u> 1.3. Let G be a finite group, EG a universal G-bundle, and $BG = * \times^G EG$ a classifying space. Then the composite map

$$A(*) \xrightarrow{\text{inclusion}} A(BG) \xrightarrow{\text{transfer}} A(EG) \simeq A(*)$$

is given by multiplication with the order of G, in the sense of the additive H-space structure.

<u>Proof</u>. We give two proofs. The first proof uses spaces with group action. The 'inclusion' map $A(*) \to A(BG)$ is induced from the exact functor

$$\begin{array}{rcl} \mathcal{R}(*) & \longrightarrow & \mathcal{U}(G) \\ Y & \longmapsto & G_+ \wedge Y . \end{array}$$

Its composite with the transfer is then simply the same map, but considered as a map to $\mathcal{R}(*)$, that is, the composite map is given by smash product with the discrete set G_+.

The second proof uses spaces over a space. As a general remark, if $\tilde{X}' \to X'$ is a fibre bundle whose fibre is of finite type, and if $X \to X'$ is any map, the resulting pullback diagram induces a commutative diagram

$$\begin{tikzcd} A(X \times_{X'} \tilde{X}') \arrow[r] & A(\tilde{X}') \\ A(X) \arrow[r] \arrow[u] & A(X') \arrow[u] \end{tikzcd}$$

in which the vertical arrows are transfers. In particular therefore we have a commutative diagram

$$\begin{array}{ccc} A(EG \times_{BG} EG) & \longrightarrow & A(EG) \\ \uparrow & & \uparrow \\ A(EG) & \longrightarrow & A(BG) \end{array} \quad .$$

The composite through the lower right is the map of the lemma, essentially. On the other hand, $EG \times_{BG} EG$ is isomorphic to the disjoint union of EG with itself indexed by the elements of G. Thus the composite map through the upper left is given by the corresponding sum of the identity map on $A(EG)$ with itself. □

We end this review by a discussion of pairings. Pairings in the algebraic K-theory of spaces can be constructed in a context of group completion [10], but it is perhaps more satisfactory to treat them in the general context of categories with cofibrations and weak equivalences.

We shall need to know a feature of the basic construction that it shares with, say, the group completion construction. Namely it is possible to iterate the construction, in a sense, but the iteration does not really produce anything new. Specifically, if $(\mathcal{C},..)$ is a category with cofibrations and weak equivalences, one can write down a certain bisimplicial category $wS.S.\mathcal{C}$. But by the additivity theorem there are homotopy equivalences

$$wS.(S_n\mathcal{C}) \xrightarrow{\sim} (wS.\mathcal{C})^n \ ,$$

hence

$$N_\Gamma(wS.\mathcal{C}) \xrightarrow{\sim} wS.S.\mathcal{C} \ ,$$

and consequently

$$|wS.\mathcal{C}| \xrightarrow{\sim} \Omega|N_\Gamma(wS.\mathcal{C})| \xrightarrow{\sim} \Omega|wS.S.\mathcal{C}|$$

since the H-space $wS.\mathcal{C}$ is group-like.

Let a *bi-exact* functor of categories with cofibrations and weak equivalences mean a functor

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{B} & \longrightarrow & \mathcal{C} \\ (A\ ,\ B) & \longmapsto & A \wedge B \end{array}$$

which becomes an exact functor if one fixes either variable. That is, for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ the partial functors $A\wedge(-)\colon \mathcal{B} \to \mathcal{C}$ and $(-)\wedge B\colon \mathcal{A} \to \mathcal{C}$ are exact. The bi-exact functor induces a pairing of the weak equivalences

$$|w\mathcal{A}| \times |w\mathcal{B}| \longrightarrow |w\mathcal{C}| \ ;$$

this may be defined on the level of nerves as the map which in degree n takes the pair of sequences of weak equivalences

$$A_1 \to A_2 \to \dots \to A_n \quad , \qquad B_1 \to B_2 \to \dots \to B_n$$

to the sequence of weak equivalences in $\mathcal{C}$,

$$A_1 \wedge B_1 \longrightarrow A_2 \wedge B_2 \longrightarrow \dots \longrightarrow A_n \wedge B_n \ .$$

As this pairing takes $|w\mathcal{A}| \vee |w\mathcal{B}|$ into the basepoint (since $A \wedge 0 = 0 \wedge B = 0$) it factors through the smash product,

$$|w\mathcal{A}| \wedge |w\mathcal{B}| \longrightarrow |w\mathcal{C}| \ .$$

The point now is simply that the same construction can be made for filtered objects. There results a pairing

$$|wS.\mathcal{A}| \wedge |wS.\mathcal{B}| \longrightarrow |wS.S.\mathcal{C}|$$

and hence, by passing to loop spaces,

$$\begin{tikzcd}
\Omega|wS.\mathcal{A}| \wedge \Omega|wS.\mathcal{B}| \arrow[r, dashed] \arrow[d] & \Omega|wS.\mathcal{C}| \arrow[d, "\wr"] \\
\Omega\Omega(|wS.\mathcal{A}| \wedge |wS.\mathcal{B}|) \arrow[r] & \Omega\Omega|wS.S.\mathcal{C}|
\end{tikzcd} \ .$$

The broken arrow represents the desired pairing. Certain naturality properties are immediate from the definition, for example the fact that the diagram

$$\begin{tikzcd}
|w\mathcal{A}| \wedge |w\mathcal{B}| \arrow[r] \arrow[d] & |w\mathcal{C}| \arrow[d] \\
\Omega|wS.\mathcal{A}| \wedge \Omega|wS.\mathcal{B}| \arrow[r] & \Omega|wS.\mathcal{C}|
\end{tikzcd}$$

commutes up to homotopy.

In the case of the algebraic K-theory of spaces, we can obtain a pairing $A(X) \wedge A(X') \to A(X\times X')$ from the smash product pairing $\mathcal{U}(G) \times \mathcal{U}(G') \to \mathcal{U}(G\times G')$ where G and G' are appropriate loop groups. Alternatively we could use the bi-exact functor $\mathcal{R}(X) \times \mathcal{R}(X') \to \mathcal{R}(X\times X')$ given by fibrewise smash product. We omit the verification that the resulting pairings are the same, up to homotopy.

Other pairings are sometimes of interest. The bi-exact functor

$$\mathcal{U}(\Sigma_m) \times \mathcal{U}(\Sigma_n) \longrightarrow \mathcal{U}(\Sigma_{m+n})$$

$$(Y , Z) \longmapsto \Sigma_{m+n} \underset{\Sigma_m\times\Sigma_n}{\curlywedge} (Y\wedge Z) \qquad (= \Sigma_{m+n} \times_{\Sigma_m\times\Sigma_n} (Y\wedge Z) \ / \ \Sigma_{m+n} \times_{\Sigma_m\times\Sigma_n} *)$$

induces a pairing $A(B\Sigma_m) \wedge A(B\Sigma_n) \to A(B\Sigma_{m+n})$ which we refer to as the *juxtaposition pairing*. It is related, of course, to the former pairing, and may be expressed in terms of it as the composite map

$$A(B\Sigma_m) \wedge A(B\Sigma_n) \longrightarrow A(B\Sigma_m \times B\Sigma_n) \longrightarrow A(B\Sigma_{m+n})$$

where the map on the right is induced from the map $B\Sigma_m \times B\Sigma_n \to B\Sigma_{m+n}$ given by juxtaposition of permutations.

2. The operations. Let P^n denote the n-th power map which takes a pointed set X to the n-fold smash product

$$P^n X = \underbrace{X \wedge \ldots \wedge X}_{\longleftarrow n \longrightarrow}$$

regarded as a Σ_n-set by permutation of the factors. We denote $P^n_j X$ the subset of $P^n X$ of the tuples involving at most j distinct elements different from the base-point. P^n_j is functorial for maps, not just isomorphisms, so we can extend to simplicial sets by degreewise extension; that is, if X is a pointed simplicial set we let $(P^n_j X)_k = P^n_j(X_k)$.

Definition. $\theta^n X = P^n X \,/\, P^n_{n-1} X$.

In other words, $\theta^n X$ is the maximal quotient of the n-fold smash product $X\wedge\ldots\wedge X$ which is Σ_n-free (in the pointed sense).

Lemma 2.1. The functor θ^n respects weak homotopy equivalences.

Proof. We show more generally that each of the functors

$$X \longmapsto P^n X \,/\, P^n_j X$$

has this property. We proceed by induction on j , starting from the case $j = 0$ which is clear. If $F_1 \to F_2$ is a cofibration of functors each of which respects weak homotopy equivalences then, by the gluing lemma, the quotient functor F_2/F_1 also respects weak homotopy equivalences. In view of this remark the inductive step from $j-1$ to j follows from the identities

$$P^n X \,/\, P^n_j X = (P^n X \,/\, P^n_{j-1} X) \,/\, (P^n_j X \,/\, P^n_{j-1} X)$$

$$P^n_j X \,/\, P^n_{j-1} X = \mathrm{Surj}(\underline{n},\underline{j}) \rtimes^{\Sigma_j} (P^j X \,/\, P^j_{j-1} X)$$

where $\underline{n}$ denotes the standard set of cardinality n , $\mathrm{Surj}(\underline{n},\underline{j})$ is the set of surjective maps from $\underline{n}$ to $\underline{j}$, and, as before, $\rtimes$ denotes the half-smash-product. □

We continue to let $\underline{n}$ denote the standard set with n elements. $\mathrm{Is}(\underline{n},\underline{k}\cup\underline{1})$ denotes the set of isomorphisms from $\underline{n}$ to $\underline{k} \cup \underline{1}$.

Lemma 2.2. There is a natural isomorphism

$$\theta^n(X \vee Y) \approx \theta^n X \vee \theta^n Y \vee \bigvee_{\substack{k+l=n \\ 0<k<n}} Is(\underline{n},\underline{k}\cup\underline{l}) \underset{\Sigma_k\times\Sigma_l}{\times} \theta^k X \wedge \theta^l Y$$

Proof. By naturality it suffices to check this in the case where X and Y are sets rather than simplicial sets. A non-basepoint element on the left hand side of the asserted equation may be identified to an injective map $\underline{n} \to X\_\cup Y\_$ where $X\_$ denotes the complement of the basepoint in X, and similarly with $Y\_$. In turn this may be identified to the equivalence class represented by a partition $\underline{n} \approx \underline{k}\cup\underline{l}$ together with a pair of injective maps $\underline{k} \to X\_$ and $\underline{l} \to Y\_$. Thus we obtain a non-basepoint element on the right, giving the desired one-one correspondence. As regards naturality (with respect to maps of pointed sets, not just their isomorphisms), the point is that as soon as any of the given description turns out to be invalid then the element in question is immediately annihilated (identified to the basepoint) and the element to which it corresponds under the isomorphism, is also annihilated. □

The maps θ^n may be assembled to one

$$\theta = \Pi_n \theta^n : \mathcal{R}(*) \longrightarrow \Pi_n \mathcal{U}(\Sigma_n) ,$$

and by lemma 2.1, θ respects weak homotopy equivalences, that is, restricts to

$$h\mathcal{R}_f(*) \longrightarrow \Pi_n h\mathcal{U}_f(\Sigma_n) .$$

It is convenient to rewrite the latter as a map (keeping the notation θ^n for the restricted maps)

$$\theta = \Pi_{n\geq 0} \theta^n : h\mathcal{R}_f(*) \longrightarrow 1 \times \Pi_{n\geq 1} h\mathcal{U}_f(\Sigma_n)$$

where θ^o is the trivial map with value 1, the multiplicative unit of $\mathcal{U}(\Sigma_o) = \mathcal{R}(*)$. Lemma 2.2 may now be restated to say that the map θ is one of H-spaces if the right hand side is equipped with the multiplication arising from the bi-exact functor

$$\Pi_n \mathcal{U}(\Sigma_n) \times \Pi_n \mathcal{U}(\Sigma_n) \longrightarrow \Pi_n \mathcal{U}(\Sigma_n)$$

$$(X_o, X_1, \ldots) , (Y_o, Y_1, \ldots) \longmapsto (Z_o, Z_1, \ldots)$$

$$Z_n = \bigvee_{k+l=n} Is(\underline{n},\underline{k}\cup\underline{l}) \underset{\Sigma_k\times\Sigma_l}{\times} X_k \wedge Y_l$$

From this bi-exact functor we obtain a pairing

$$(\Pi_n A(B\Sigma_n)) \wedge (\Pi_n A(B\Sigma_n)) \longrightarrow \Pi_n A(B\Sigma_n) .$$

(In fact, even though the map

$$\Omega|hS.(\Pi_n \mathcal{U}_f(\Sigma_n))| \longrightarrow \Pi_n \Omega|hS.\mathcal{U}_f(\Sigma_n)|$$

is only a weak homotopy equivalence, we can get the pairing well-defined, not just well-defined up to weak homotopy. For the corresponding map for finite products is an isomorphism. So there are well-defined pairings involving the finite products indexed by $0,1,\ldots,m$, and from these we can get the pairing as stated, by inverse limit).

From the pairing we obtain a multiplicative H-space structure on

$$1 \times \Pi_{n\geq 1} A(B\Sigma_n) \quad .$$

<u>Lemma</u> 2.3. The H-space $1 \times \Pi_{n\geq 1} A(B\Sigma_n)$ is group-like.

<u>Proof</u>. If M is any space, and f a homotopy class of maps,

$$f \in [\, M \, , \, 1 \times \Pi_{n\geq 1} A(B\Sigma_n) \,] \quad ,$$

let us write

$$f = (\, 1 \, , \, f_1 \, , \, f_2 \, , \, \ldots \,)$$

where $f_n \in [M, A(B\Sigma_n)]$. The multiplication of such series is given by

$$(fg)_n = \sum_{k+l=n} f_k \, g_l$$

where $f_k \, g_l$ denotes the composite map

$$M \xrightarrow{\text{diag}} M \times M \xrightarrow{f_k \times g_l} A(B\Sigma_k) \times A(B\Sigma_l) \longrightarrow A(B\Sigma_{k+l}) \quad .$$

The neutral element is the series $(1,0,0,\ldots)$, and the inverse of $(1,f_1,f_2,\ldots)$ may be obtained by inductively solving the equations

$$0 = g_n + \sum_{\substack{k+l=n \\ 0<k\leq n}} f_k \, g_l$$

using the fact that the additive H-space structure on $A(B\Sigma_n)$ is group-like. □

<u>Proposition</u> 2.4. There is a map of H-spaces, unique up to weak homotopy,

$$A(*) \longrightarrow \Pi_n \, A(B\Sigma_n)$$

and denoted θ again, by abuse of notation, having the properties that $\theta^0 = 1$, $\theta^1 = \mathrm{id}$, and that the diagram

$$\begin{array}{ccccc}
hR^m(*) & \longrightarrow & hR_f(*) & \longrightarrow & \Pi_n \, hU_f(\Sigma_n) \\
\big\downarrow & & & & \big\downarrow \\
hR_f(*) & \longrightarrow & A(*) & \longrightarrow & \Pi_n \, A(B\Sigma_n)
\end{array}$$

commutes up to weak homotopy, for every m.

Proof. This results by application of the universal property of lemma 1.2 to the map of H-spaces given by the composition

$$hR_f(*) \longrightarrow 1 \times \Pi_{n\geqslant 1}\, hU_f(\Sigma_n) \longrightarrow 1 \times \Pi_{n\geqslant 1}\, A(B\Sigma_n) \quad .$$

The relevant hypotheses are checked in lemma 2.3 above and lemma 2.5 below. □

Lemma 2.5. For every n , the two composite maps

$$hS_2R_f(*) \overset{t}{\underset{svq}{\rightrightarrows}} hR_f(*) \xrightarrow{\theta^n} hU_f(\Sigma_n) \longrightarrow A(B\Sigma_n)$$

are homotopic.

Proof. We use

Sublemma. To a cofibration $W \rightarrowtail X$ in $R(*)$ there is canonically associated a filtration $Y_n \rightarrowtail Y_{n-1} \rightarrowtail \dots \rightarrowtail Y_o$, with $Y_o = \theta^n X$, together with isomorphisms

$$Y_k \,/\, Y_{k+1} \approx \operatorname{Is}(\underline{n},\underline{k}\cup\underline{l}) \underset{}{\overset{\Sigma_k\times\Sigma_l}{\times}} (\theta^k W \wedge \theta^l(X/W)) \qquad \text{(where } l=n-k) \ .$$

To deduce the lemma from the sublemma we apply lemma 1.1 to the canonical filtration $Y_n \rightarrowtail \dots \rightarrowtail Y_o$. We obtain that the composition of the map $\theta^n\, t$,

$$(W \rightarrowtail X) \longmapsto \theta^n X \quad ,$$

with the map $hU_f(\Sigma_n) \to A(B\Sigma_n)$, is homotopic to the composition of the latter with

$$(W \rightarrowtail X) \longmapsto Y_n \vee Y_{n-1}/Y_n \vee \dots \vee Y_o/Y_1$$

which is θ^n (svq) in view of lemma 2.2 and the isomorphisms of the sublemma.

It remains to prove the sublemma. The term Y_k in the filtration is defined as the simplicial subset of $\theta^n X$ involving tuples with at least k elements in W . To establish the asserted isomorphisms it suffices, by naturality, to treat the case where X is a set rather than simplicial set. A non-basepoint element of $\theta^n X$ may then be identified, as before, to an injective map into the complement of the basepoint, $\underline{n} \to X_-$. If the element is in Y_k , but not in Y_{k+1} , the associated map takes precisely k elements into W . The element may thus be identified to the equivalence class represented by a partition $\underline{n} \approx \underline{k}\cup\underline{l}$ together with a pair of injective maps $\underline{k} \to W_-$ and $\underline{l} \to (X/W)_-$, it therefore corresponds to a non-basepoint element on the right. In checking the naturality of this isomorphism we must take into account that a map $(W \rightarrowtail X) \longrightarrow (W' \rightarrowtail X')$ will not take the complement of W in X into the complement of W' in X' , in general. The effect of this is that certain (extra) elements are annihilated by the induced maps of the left and right terms of the equation. But such elements correspond under the isomorphism.

To validate the application of the sublemma, we should also show that the functor

$(W \rightarrowtail X) \mapsto Y_k$ respects weak homotopy equivalences. This is a verification along the lines of lemma 2.1, but more complicated. We bypass this verification, replacing it by the following argument. It is certainly true that $(W \rightarrowtail X) \mapsto Y_k$ takes weak homotopy equivalences to *homology equivalences* (by excision and the isomorphisms of the sublemma this follows from lemma 2.1). This now suffices for the purpose of the lemma. The reason is that $A(B\Sigma_n)$ may also be defined in terms of the somewhat larger category of weak equivalences $h_Z\mathcal{U}_f(\Sigma_n)$, the maps inducing isomorphisms in homology with integral coefficients. Indeed, the exact functor of $\mathcal{U}_f(\Sigma_n)$ to itself given by double suspension, induces endomorphisms of both $h_Z S.\mathcal{U}_f(\Sigma_n)$ and $hS.\mathcal{U}_f(\Sigma_n)$ which are homotopic to the respective identity maps, and it takes the former into the latter, by the Whitehead theorem; thus the inclusion $hS.\mathcal{U}_f(\Sigma_n) \to h_Z S\, \mathcal{U}_f(\Sigma_n)$ is a homotopy equivalence. □

We shall need in a moment

Lemma 2.6. The product $\prod_{n\geq 1} A(*)_{(n)}$, where $A(*)_{(n)} = A(*)$, can be given a composition law so that the map

$$\prod_n A(B\Sigma_n) \longrightarrow \prod_n A(*)_{(n)}$$

whose components are the transfer maps $A(B\Sigma_n) \to A(*)$, is a map of H-spaces.

Proof. The composition law is induced from the bi-exact functor

$$\prod_{n\geq 0} R(*)_{(n)} \times \prod_{n\geq 0} R(*)_{(n)} \longrightarrow \prod_{n\geq 0} R(*)_{(n)}$$

$$(X_0, X_1, \ldots) \ , \ (Y_0, Y_1, \ldots) \longmapsto (Z_0, Z_1, \ldots)$$

$$Z_n = \bigvee_{k+l=n} (\Sigma_{k+l} \times^{\Sigma_k \times \Sigma_l} *) \wedge (X_k \wedge Y_l) \ .$$

The asserted compatibility of composition laws is simply the fact that the diagrams

$$X\ ,\ Y \longmapsto \Sigma_{k+l} \wedge^{\Sigma_k \times \Sigma_l} X \wedge Y$$

$$\begin{array}{ccc} \mathcal{U}(\Sigma_k) \times \mathcal{U}(\Sigma_l) & \longrightarrow & \mathcal{U}(\Sigma_{k+l}) \\ \downarrow & & \downarrow \\ R(*) \times R(*) & \longrightarrow & R(*) \end{array}$$

$$X\ ,\ Y \longmapsto (\Sigma_{k+l} \times^{\Sigma_k \times \Sigma_l} *) \wedge X \wedge Y$$

commute, where the vertical arrows are given by forgetting the action. □

<u>Proposition</u> 2.7. The composition of the operation θ^n with the transfer map,

$$A(*) \xrightarrow{\theta^n} A(B\Sigma_n) \xrightarrow{\phi_n} A(*) \ ,$$

is the same, up to weak homotopy, as the polynomial map on $A(*)$ given by the polynomial $x(x-1)\dots(x-n+1)$.

<u>Proof</u>. In view of the preceding lemma, the map $A(*) \longrightarrow \Pi_n A(B\Sigma_n) \longrightarrow \Pi_n A(*)_{(n)}$ is one of H-spaces, so the uniqueness clause of the universal property (lemma 1.2) applies, and to show the map equals a certain other map, up to weak homotopy, it will therefore suffice to make that comparison after composition with $h\mathcal{R}_f(*) \to A(*)$. Alternatively, in view of the defining property of θ (proposition 2.4) it suffices to show that the composite map

$$h\mathcal{R}_f(*) \longrightarrow \Pi_n\ h\mathcal{U}_f(\Sigma_n) \longrightarrow \Pi_n\ h\mathcal{R}_f(*)_{(n)} \longrightarrow \Pi_n\ A(*)_{(n)}$$

may be re-expressed in terms of polynomial maps in the asserted way.

The polynomials $p_n(x) = x(x-1)\dots(x-n+1)$ can be recursively defined in terms of the identity

$$(n-1)\ p_{n-1}(x) + p_n(x) = p_{n-1}(x)\ x \ ,$$

the asserted comparison will therefore be established once we show that the maps

$$h\mathcal{R}_f(*) \xrightarrow{\phi_n\theta^n} h\mathcal{R}_f(*) \longrightarrow A(*)$$

satisfy a similar identity, up to homotopy.

In view of lemma 1.1, the required homotopy

$$\underline{(n-1)}\lambda\phi_{n-1}\theta^{n-1} \vee \phi_n\theta^n \simeq \phi_{n-1}\theta^{n-1}\wedge\theta^1$$

will be implied by a cofibration sequence of functors $\mathcal{R}(*) \to \mathcal{R}(*)$,

$$\underline{(n-1)}\lambda\phi_{n-1}\theta^{n-1} \rightarrowtail \phi_{n-1}\theta^{n-1}\wedge\theta^1 \twoheadrightarrow \phi_n\theta^n \ .$$

To establish the existence of that cofibration sequence it suffices, by naturality, to treat the case where X is a pointed set rather than simplicial set. $\phi_n\theta^n X$ is obtained from $\phi_{n-1}\theta^{n-1}X \wedge X$ by discarding those elements which are represented by non-injective maps

$$\underline{n-1} \cup \underline{1} \longrightarrow X_- \ ,$$

and for every non-basepoint element of $\phi_{n-1}\theta^{n-1}X$ there are precisely $n-1$ such non-injective maps, depending on where the extra element 1 is being mapped; or re-expressed functorially, the set of those maps is obtained by half-smash-product with the set $\underline{(n-1)}$. □

We conclude with a brief discussion of generalizations and variants.

The operation θ^n may be regarded as the special case $X = *$ of a map

$$A(X) \longrightarrow A(D_n X)$$

where

$$D_n X = E\Sigma_n \times^{\Sigma_n} X^n$$

(the bundle over the classifying space of Σ_n associated to the permutation representation of Σ_n on the factors of the cartesian product X^n). These more general operations also satisfy the 'Cartan formula' in the sense that $A(X) \longrightarrow \prod_{n\geq 1} A(D_n X)$ is a map of H-spaces if the right hand side is equipped with the composition law arising from the juxtaposition pairings $A(D_m X) \times A(D_n X) \to A(D_{m+n} X)$. It is not clear on the other hand what, if any, takes the role of proposition 2.7.

The elementary construction of the θ^n is compatible not just with weak homotopy equivalences (lemma 2.1), but also with other types of weak equivalences. Specifically it is compatible with the *rational homology equivalences*. It may be shown that $\Omega|h_{\mathcal{Q}}S.\mathcal{U}_f(G)|$ is the same, up to homotopy, as $K(\mathcal{Q}[G])$, the algebraic K-theory of the rational group ring $\mathcal{Q}[G]$. Thus one obtains operations $\theta^n\colon K(\mathcal{Q}) \to K(\mathcal{Q}[\Sigma_n])$. The analogue of proposition 2.7 is true for these operations, that is, the composite of θ^n with the transfer $K(\mathcal{Q}[\Sigma_n]) \to K(\mathcal{Q})$ may be re-expressed as a polynomial map, in the same way.

A variant of the construction may be used to construct the exterior power operations in the algebraic K-theory of a commutative ring R . This corresponds, on the elementary level, to the possibility of taking a projective R-module P to its n-th tensor power $P\otimes_R \ldots \otimes_R P$ and then decomposing this suitably. It is not in general possible, however, to extract from $P^{\otimes n}$ its 'Σ_n-free part', a module which is projective over $R[\Sigma_n]$ rather than just R . Thus the method fails to provide operations $K(R) \to K(R[\Sigma_n])$, in general.

Indeed, not just the method, even the result seems to fail in general. Specifically in the case $R = \mathcal{Z}$, the ring of integers, there cannot exist operations $K(\mathcal{Z}) \to K(\mathcal{Z}[\Sigma_n])$ which satisfy the analogue of proposition 2.7. For their existence would imply, as in the introduction, that for every prime p the transfer map $\tilde{K}_j(\mathcal{Z}[\Sigma_p])_{(p)} \to K_j(\mathcal{Z})_{(p)}$ is surjective for $j > 0$. But this is not true. In particular, the transfer map $\tilde{K}_3(\mathcal{Z}[\Sigma_2]) \to K_3(\mathcal{Z})$ is not surjective on the 2-torsion. To see this, let

$$\varepsilon,\ \delta\colon\ \mathcal{Z}[\Sigma_2] \longrightarrow \mathcal{Z}$$

denote the two ring homomorphisms given by the augmentation and by the identification of Σ_2 with the group of units of $\mathcal{Z}$, respectively. Let $\mathcal{Z}'$ be obtained from $\mathcal{Z}$ by inverting 2 . The map

$$\varepsilon' \times \delta' : Z'[\Sigma_2] \longrightarrow Z' \times Z'$$

is an isomorphism of rings, so the transfer map $K(Z'[\Sigma_2]) \to K(Z')$ may be identified to the sum of $\varepsilon^!_*$ and $\delta^!_*$. The augmentation map $\varepsilon^!_*$ is trivial on the reduced part $\tilde{K}(Z'[\Sigma_2]) = \mathrm{fibre}(\ K(Z'[\Sigma_2] \to K(Z')\ )$, so the transfer map may be identified to $\delta^!_*$ on that part. In view of theorems of Quillen (the localization theorem and the computation of the K-theory of finite fields) the map $K_3(Z) \to K_3(Z')$ is an isomorphism on the 2-torsion. We compare the two diagrams

$$\begin{tikzcd} \tilde{K}_3(Z[\Sigma_2]) \arrow[r] \arrow[d, "(\mathrm{transfer})"', "(\delta_*)"] & \tilde{K}_3(Z'[\Sigma_2]) \arrow[d, "\delta^!_* = \mathrm{transfer}"] \\ K_3(Z) \arrow[r, "\approx_{(2)}"'] & K_3(Z') \end{tikzcd}$$

where the arrow on the left can be either the transfer or δ_*, respectively. If the transfer were surjective on the 2-torsion, we could conclude from this comparison that the map $\delta_*: \tilde{K}_3(Z[\Sigma_2]) \to K_3(Z)$ were also surjective on the 2-torsion. But this is false, as was shown by Guin-Walery and Loday [2] as a consequence of the Lee-Szczarba computation of $K_3(Z)$ and of work of their own on excision.

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